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ITERATIVE METHODS FOR SOLVING AN EQUILIBRIUM PROBLEM ARISING IN TRANSIT DEREGULATION

by

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ABSTRACT

We consider the context of a deregulated transit system involving private cars, bus passengers, and bus operators, where the latter freely decides which line it operates. We assume that car drivers and transit users strive to minimize individual travel times whereas bus operators maximize individual profits. Within each system under consideration – cars, passengers, buses– a state of equilibrium can be characterized as the solution of a variational inequality. In this paper, we assume that the combined model is solved by the Gauss–Seidel approach described in Fernández and Marcotte [1988]. Special attention is paid to the solution of the bus operators equilibrium for which several solution algorithms– Jacobi–Newton, Newton–Jacobi, fixed point iterations– are proposed. Numerical results are presented for a small network and for the Santiago (Chile) transit network.

Key words : Network Equilibrium, Transit Deregulation, Variational Inequalities.

RÉSUMÉ

Nous considérons un modèle d'équilibre multimodal du trafic urbain provenant d'un contexte déréglementé où chaque conducteur d'autobus choisit son trajet. Alors que les automobilistes et les usagers du transport en commun se répartissent sur le réseau de facon à minimiser leur temps de transport individuel, les conducteurs d'autobus maximisent leur profit (individuellement). A l'équilibre, chaque sous–système (automobiles, passagers, autobus) peut être représenté à l'aide d'un ensemble de variables, qui satisfait une inéquation variationnelle. Fernández et Marcotte [1988] ont récemment proposé une approche itérative de type Gauss–Seidel pour la résolution du modèle global. Dans cet article nous nous concentrons sur la résolution du problème d'équilibre des opérateurs pour lequel nous proposons plusieurs algorithmes itératifs: Jacobi–Newton, Newton–Jacobi, algorithmes de point fixe (itération de Picard). Les résultats numériques présentés ont été obtenus à partir d'un réseau fictif ainsi que du réseau de transport en commun de la ville de Santiago (Chili).

Mots Clés : équilibre de réseau, déréglementation–transport en commun, inéquations variationnelles.

1. Introduction.

Transit deregulation has been a controversial issue ever since the ediction of the 1924 London Traffic Act giving power to the Minister of Transport to impose upper limits for the number of buses operating on main roads of the metropolitan area. Consequently, new competition was effectively prohibited and monopoly rights confered to a small group of companies, with the purpose of reducing congestion. Later on, licensing systems were introduced in order to control various aspects of the traffic environment –prices, service level, etc.– and to prevent unacceptable practices (see Banister [1985]).

In most western urban areas, the public transportation service has been under the control of a centralized transit authority responsible for the planning and operation of the transit system. Subsidies necessary to operate the system under regulated conditions have progressively become a burden for local governments (see Fernández and De Cea, 1985) thus leading to a renewed interest in transit deregulation. We can quote as examples the bus deregulation introduced in Great Britain with the Transport Act of 1985 and the experiment conducted in the city of Santiago de Chile where transit deregulation has been in effect since 1978.

As the concept of deregulation becomes increasingly "fashionable", one observes a growing interest among the traffic planners and the research community for analytical tools allowing a better understanding of the mechanism of deregulation (see Beesley and Glaister [1985], Foster [1985], Gwilliam et al. [1985], and White and Turner [1985]).

Recently, based on the Santiago experience, Fernández and Marcotte [1988] proposed a traffic equilibrium model involving cars, transit users and bus operators within a deregulated context where each individual bus driver can decide which line he is going to operate. In this model it is assumed that car drivers and passengers behave according to Wardrop's first principle of traffic equilibrium ("user equilibrium") while bus operators strive to maximize individual profits. For each subsystem–cars, passengers, operators– under consideration, there is a (not necessarily unique) equilibrium state that can be characterized as the solution set associated with a variational inequality. In this paper we present and give numerical results for iterative algorithms that solve the bus-operators subproblem arising in the first formulation of the model proposed by Fernández and Marcotte [1988]. Assuming that car and passenger flows and the fleet size are exogenously given (there is no free entry or exit from the system) , the algorithms assign the buses among the lines of the underlying network so that individual profits are maximized. If the frequencies thus obtained are consistent with the flows obtained from a bimodal car-passengers assignment model, then an equilibrium for the global model has been achieved. Otherwise the new frequency vector is used as an input for a new bimodal assignment, thus forming one iteration of the global Gauss–Seidel iterative scheme described in Fernández and Marcotte [1988].

The paper is organized as follows: Section 2 sketches the bus–operators model, solution algorithms are presented in Section 3; finally, Section 4 presents numerical results obtained from (i) a small network composed of 29 nodes and 8 bus lines (ii) the transit network of the city of Santiago de Chile.

2. The Bus-Operators Model.

Consider a transit system where operators profits are expressed as functions of car flows (fixed), passenger flows and line frequencies. Vehicles operating cost are assumed to be dependent on traffic conditions (see Fernández and Marcotte [1988] for a complete description of the cost functions).

Denote by L the number of lines in the system. Throughout the paper we will adopt the following notation:

$\mathcal{B}_l$ = number of buses on line l, $\quad l = 1, \ldots L$;

$\mathcal{B} = (\mathcal{B}_1, \ldots \mathcal{B}_L)^t$

$\mathcal{F}_l$ = total number of boarding passengers per unit of time on line l;

FC_l = fixed operating cost per unit of time on line l;

$G_l(\Phi)$ = variable operating cost on line l;

ρ_l = bus fare on line l;

ϕ_l = bus frequency on line l;

$\Phi = (\phi_1, \ldots, \phi_L)$ vector of frequencies;

$T_l(\Phi) =$ round trip travel time for a bus on line l;

The profit function for line l then takes the form:

$$\Pi_l(\Phi) = \frac{\rho_l \mathcal{F}_l}{\mathcal{B}_l} - \frac{G_l(\Phi)}{T_l(\Phi)} - FC_l \quad l = 1, \ldots, L$$

Here we recall that car and passengers flows are exogenous. Dependency of profit on car flows and congestion is taken into account through the monotonically increasing functions $T_l(\Phi)$ and $G_l(\Phi)$.

In order to simplify notation, we denote by $\Psi(\Phi)$ the total cost function expressed in dollars per unit of time:

$$\Psi(\Phi) = \frac{G_l(\Phi)}{T_l(\Phi)} - FC_l, \quad l = 1, \ldots, L$$

After setting $\alpha_l = \rho_l \mathcal{F}_l$ we express the profit function as total revenue minus total cost:

$$\Pi_l(\mathcal{B}) = \frac{\alpha_l}{\mathcal{B}_l} - \Psi_l(\Phi), l = 1, \ldots, L. \tag{1}$$

The line frequency ϕ_l and the fleet size $\mathcal{B}_l$ are related by the expression:

$$\phi_l = \frac{\mathcal{B}_l}{T_l(\Phi)} \tag{2}$$

where $T_l(\Phi)$ stands for the round trip travel time for a bus operating on line l. This relationship is then used to express the profit function (1) in terms of frequencies only:

$$\Pi_l(\Phi) = \frac{\alpha_l}{\phi_l T_l(\Phi)} - \Psi_l(\Phi), \quad l = 1, \ldots, L. \tag{3}$$

It is easy to see that the equilibrium state for the bus operators system will be achieved when operators profits are equalized. If the number of buses in the system is not restricted —free entry— equilibrium profits will be zero, i.e. $\Pi_l(\Phi) = 0, l = 1, \ldots, L$. But in presence of the fleet size constraint

$$\sum_{l=1}^{L} \mathcal{B}_l = \mathcal{B}$$

the profit μ may be positive (low competition), zero (perfect competition), or negative (strong competition). The equilibrium vector Φ is then seen to be a solution to the non-linear complementarity problem:

$$\begin{aligned} \Pi_l(\Phi) - \mu \leq 0, \quad & l = 1, \ldots, L \\ \phi_l'(\Pi_l(\Phi) - \mu) = 0, \quad & l = 1, \ldots, L \\ \sum_l \phi_l T_l(\Phi) = \mathcal{B} & \\ \Phi \geq 0, & \end{aligned} \tag{4}$$

where μ stands for the profit common to all bus operators and $\mathcal{B}$ represents the total fleet size. In this model one observes from (3) that if the passenger flow on a line is zero, then the corresponding frequency will also be zero. This allows to consider only the set of lines carrying positive passenger flow . Problem (4) then reduces to the system of nonlinear equations:

$$\Pi_l(\Phi) - \mu = 0, \quad \forall l \in \mathcal{L} \tag{5}$$

$$\sum_{l \in \mathcal{L}} \phi_l T_l(\Phi) = \mathcal{B} \tag{6}$$

In the following section several approaches for solving system (5)-(6) are discussed.

3. Solution procedures

All methods discussed in this paper may be cast into a general iterative framework. Distinct algorithms differ only in the way the next iterate is determined. The general scheme has the form:

STEP 0. Initialize μ^0 and Φ^0; k:=0.

STEP 1. Determine μ^{k+1} and Φ^{k+1}

STEP 2. If stopping criteria is met then stop; else k:=k+1 and goto STEP 1.

We now present different methods for performing STEP 1.

3.1. Estimation of μ^{k+1}.

The main idea consists in expressing the common profit μ as a function of the frequency vector Φ. We present two ways of achieving this. First, substituting eqn(3) in eqn(5) and summing over all lines one obtains:

$$\sum_{l \in \mathcal{L}} \frac{\alpha_l}{\mu + \Psi_l(\Phi)} = \sum_{l \in \mathcal{L}} \phi_l T_l(\Phi) = \mathcal{B}$$

It seems natural then to define μ^{k+1} as the solution to the non-linear equation: (see Fig.1)

$$h(\mu) = \sum_{l \in \mathcal{L}} \frac{\alpha_l}{\mu + \Psi_l(\Phi^k)} = \mathcal{B} \tag{7}$$

This equation can be solved by using a first order approximation (Newton's method), thus yielding:

$$\sum_{l \in \mathcal{L}} \frac{\alpha_l}{\Psi_l(\Phi^k)}\left[1 - \frac{\mu^{k+1}}{\Psi_l(\Phi^k)}\right] = \mathcal{B}$$

$$\text{and} \quad \mu^{k+1} = \left(\sum_l \frac{\alpha_l}{\Psi_l(\Phi^k)} - \mathcal{B}\right) \Big/ \left(\sum_l \frac{\alpha_l}{[\Psi_l(\Phi^k)]^2}\right).$$

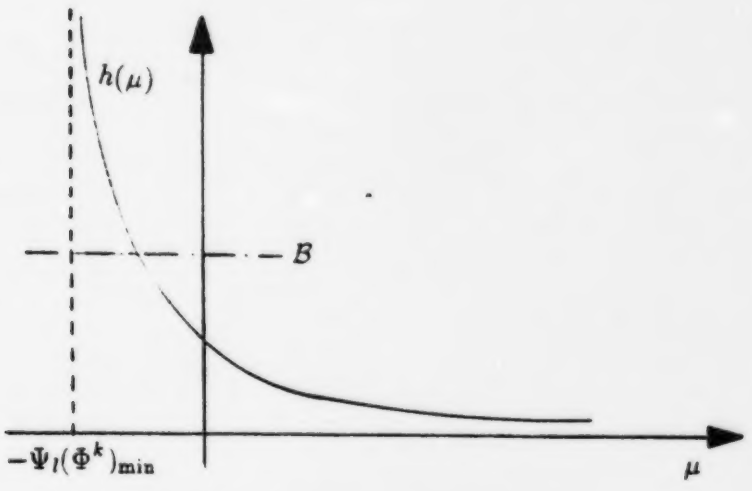

Figure 1. General form of equation (7).

It is also possible, at the expense of slightly more involved calculations, to use a second order approximation. In either case the solution must satisfy:

$$\mu^{k+1} > -\Psi_l(\Phi^k)_{\min} = \min_l\{\Psi_l(\Phi^k)\}.$$

A second way of expressing μ is obtained from the relation:

$$\phi_l T_l(\Phi) = \frac{\alpha_l - \phi_l T_l(\Phi)\Psi_l(\Phi)}{\mu}. \tag{8}$$

Adding over all lines and solving for μ there comes:

$$\mu = -FC_l + \frac{1}{B}\sum_l\{\alpha_l - \phi_l T_l(\Phi)\Psi_l(\Phi)\}. \tag{9}$$

It ensues that μ^{k+1} can be obtained simply by evaluating eqn(9) at the current iterate Φ^k.

At equilibrium, μ represents the profit common to all transit lines; in eqn(9) this profit is temporarily (for one iteration) set to the average profit per bus. It is also possible to set μ, at a given iteration, to the average profit per line. However this strategy proved inferior in all tests, and the corresponding numerical results are not reported here.

3.2. Estimation of Φ^{k+1}.

We propose three ways of defining the next iterate for the frequency vector Φ. The first is a fixed point iteration based on (8):

$$\phi_l^{k+1} = \frac{\alpha_l}{T_l(\Phi^k)[\Psi_l(\Phi^k) + \mu^{k+1}]}, \qquad l \in \mathcal{L} \tag{10}$$

where μ^{k+1} is defined by any of the methods described in the previous section.

Actually, there are several ways the profit function $\Pi(\Phi)$ may be utilized to design a fixed point iterative algorithm. Let us define a vector function $F=\{F_l(\Phi)\}_{l\in\mathcal{L}}$ obtained from eqn(3) by simple algebraic manipulations, and whose fixed points coincide with the solutions of the system of equations (5)–(6). Possible choices for $F_l(\Phi)$ are: (The first choice of $F_l(\Phi)$ yields the preceding fixed point iteration)

$$F_l(\Phi) = \phi_l T_l(\Phi) - \frac{\alpha_l}{\mu + \Psi_l(\Phi)} \tag{12}$$

$$F_l(\Phi) = \frac{\alpha_l}{\phi_l T_l(\Phi)} - \Psi_l(\Phi) - \mu \tag{13}$$

$$F_l(\Phi) = \phi_l[B_l T_l(\Phi) + G_l(\Phi)] - \alpha_l \tag{14}$$

$$F_l(\Phi) = \phi_l - \frac{\alpha_l}{B_l T_l(\Phi) + G_l(\Phi)} \tag{15}$$

$$F_l(\Phi) = B_l T_l(\Phi) + G_l(\Phi) - \frac{\alpha_l}{\phi_l} \tag{16}$$

where $B_l = FC_l + \mu$.[1] Each equation (12)–(16), although mathematically equivalent, yields numerically distinct fixed points algorithms, obtained by setting, at iteration k, the frequency vector Φ to the previous value Φ^{k-1} in the respective right-hand-sides of equations (12)–(16).

A second class of algorithms is derived from the linearization of equations (5)–(6), where the linearized system of equations is solved using Jacobi's method, yielding:

$$\phi_l^{k+1} = \phi_l^k - \frac{F_l(\Phi^k)}{F_l'(\Phi^k)} \qquad l \in \mathcal{L}, \tag{17}$$

where $F_l(\Phi^k)$ is defined by any of the equations (10)-(16) and F_l' denotes the partial derivative of the function F_l with respect to the variable ϕ_l. This will be referred to as a Newton–Jacobi iterative scheme.

Finally a third class of algorithms is obtained from the relation:

$$\Phi^{k+1} = \Phi^k + d^k \tag{18}$$

where d^k is a solution to the problem:

$$\nabla F(\Phi^k) \cdot d^k = -F(\Phi^k) \tag{19}$$

and ∇F denotes the Jacobian matrix of F evaluated at Φ^k. In order to avoid the full storage of the matrix $\nabla F(\Phi^k)$ that would result from a straightforward application of Newton's method, the linear system (19) is solved using a conjugate gradient method.

The previous methods of determining μ^{k+1} and Φ^{k+1} can be combined to yield 36 conceptually different algorithms. Actually several variants of these algorithms can be devised; let us illustrate some of these variants using the fixed point algorithm described by (10)[2]. At iteration k, rather than holding the right-hand-side fixed, a better approximation of ϕ_l could be obtained by either:

- linearizing the right-hand-side of (10). This corresponds to a Jacobi– Newton scheme (see Ortega and Rheinboldt [1970])
- linearizing T_l or(and) Ψ_l; then ϕ_l^{k+1} is solution of a second (or third) order polynomial equation for which there exist closed form expressions.
- making a second order approximation of the right-hand-side of (10).
- etc.

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[1] The first choice of $F_l(\Phi)$ yields the preceding fixed point iteration

[2] The techniques described below are also applicable to the fixed point algorithms defined by (12–(16).

In the next section we present numerical results corresponding to some of the most interesting combinations.

4. Numerical Results.

The algorithms have been implemented on two transit networks. The first is a small network utilized in a previous study by Fernández et al. [1986a ,1986b] composed of 29 nodes, 87 arcs and 8 bus lines (see Fig. 2 and Table 5). The second corresponds to the transit network of the city of Santiago de Chile [Dictuc, 1978] and has 245 nodes, 654 arcs and 223 bus lines.

In our implementation the algorithm has been stopped whenever

$$\max_l |\Pi_l(\Phi^k) - \mu| < \varepsilon$$

where the tolerance ε has been fixed to 10^{-5}. The code was written in PASCAL and the test performed on a VAX 11/780 at the University of Montreal, running under the operating system VAX/VMS version 4.3.

For both networks, $\mathcal{F}_l$, the total number of passengers boarding line l, was obtained from the EMME/2 transit assignment model (EMME/2, [1982] and Babin et al. [1982]). Parameters FC_l and ρ_l have been set respectively to values 0 and 1 on all lines. Travel time and operating cost functions were arc separable and assumed the general form:

$$T_l(\Phi) = \sum_a \delta_{al} t_a(\Phi)$$

$$G_l(\Phi) = \sum_a \delta_{al} g_a(\Phi)$$

where

$$t_a(\Phi) = \bar{\alpha}_a + \bar{\beta}_a (f_a + \nu\phi_a)^{\bar{n}},$$

$$g_l(\Phi) = \tilde{\alpha}_a + \tilde{\beta}_a (f_a + \nu\phi_a)^{\tilde{n}} + \gamma_a \hat{f}_a^l$$

$\bar{\alpha}_a$ = free flow travel time on arc a

$\tilde{\alpha}_a$ = free flow operating cost on arc a

$\tilde{\beta}_a, \bar{\beta}_a$ = congestion parameters on arc a

f_a = car flow on arc a

ν = car equivalent of one bus (for congestion purposes)

ϕ_a = total frequency on arc a

$\bar{n}, \tilde{n}$ = congestion parameters

$\hat{f}_a^l$ = nb. of passengers boarding line l at the head of arc a

γ_a = delay due to boarding on arc a

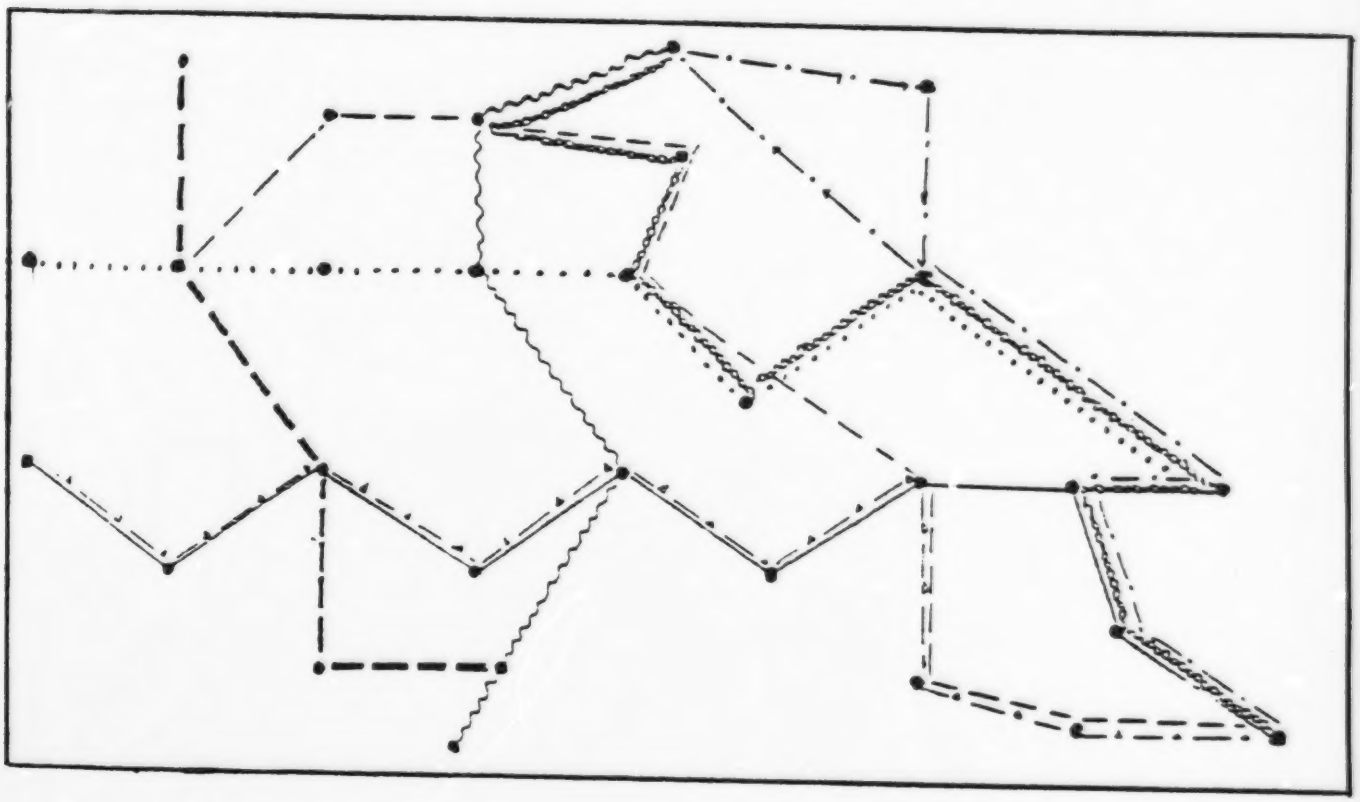

Figure 2. Small network example.

$\mathcal{B}$	Estimation of μ^{k+1}			
	$\mu^{(1)}$		$\mu^{(2)}$	
	# iter.	time[3]	# iter.	time[3]
100	3	220	5	270
500	5	140	13	330
1000	7	160	74	1470
2000	11	200	N.C.	N.C.
5000	N.C.	N.C.	N.C	N.C.

$\mathcal{B}$ = fleet size; N.C.: time limit exceeded
(1): Newton's method applied to eqn(7)
(2): estimated from eqn(9)
(3): microseconds.

Table 1. Comparison of two Fixed Point Iterations.

Numerical results for the small example are summarized in tables 1–3. CPU times are reported in microseconds. Table 1 shows the results of the fixed point iterations when the profit μ is estimated using either Newton's method (eqn(7)) or eqn(9). Whenever $\mathcal{B}$ exceeds the values 3000, convergence is very slow. The best performance was obtained using eqn(9), and problems with high values of the fleet size $\mathcal{B}$ proved increasingly difficult to solve.

In table 2 we notice the slow convergence rate of the Newton-Conjugate Gradient method. Running time varies significantly according to the expression for the profit function being used. For this case, and for small values of $\mathcal{B}$, the best performances have been obtained using expressions (12) and (14) . However, the expressions became unstable as the value of the fleet size increased.

Table 3a and 3b present the results for the Newton–Jacobi iterates for which the best results have been obtained. All but one method (based on eqn(12)) converged well over a wide range of values of the total fleet size $\mathcal{B}$, with slight differences in CPU times.

In all experiments corresponding to tables 2, 3a and 3b, the value of μ^k has been obtained by solving eqn(7) very accurately using Newton's method. We also tested a variant where a single Newton iteration was performed, hoping to lower the CPU time. This "low accuracy" variant proved competitive with the original version, although CPU time was not significantly decreased.

Another strategy tested was, in eqn(12), to linearize the nonlinear function $T_l(\Phi)$ but not the ratios $\frac{\alpha_l}{\phi_l}$, resulting in a quadratic equation in the variable ϕ_l. Again, this variant although competitive, would not decrease CPU time significantly.

$\mathcal{B}$	Profit Function							
	eqn(12)		eqn(13)		eqn(14)		eqn(16)	
	#iter.	time[1]	#iter.	time[1]	#iter.	time[1]	#iter.	time[1]
1000	6	420	9	640	6	400	9	760
2000	8	530	11	720	8	570	9	840
3000	10	700	13	770	10	680	14	1030
4000	16	1090	14	810	15	1060	16	1090
5000	21	1580	15	930	-	N.C.	22	1440
6000	23	1780	17	1040	-	N.C.	27	1760
7000	-	N.C.	20	1170	-	N.C.	33	2120
8000	-	N.C.	24	1440	-	N.C.	37	2370
9000	-	N.C.	27	1650	-	N.C.	42	2680
10000	-	N.C.	29	1810	-	N.C.	46	2760

$\mathcal{B}$ = fleet size; N.C.: time limit exceeded
[1]: microseconds.

Table 2. Newton–Conjugate Gradient method on small network.

Finally, table 4 presents numerical results for the Santiago data, using either eqn(15) or (16) (which performed the best on the small example) for the profit function, and estimators (7) or (9) for μ. The best combination for this example proved to be eqn(9) for μ with eqn(16) for the profit function; remark that bus-induced congestion is so large for high values of $\mathcal{B}$ ($\mathcal{B} > 5000$) that, as a consequence, travel times are extended beyond reasonable limits. This results in the following undesirable side-effects: extension of the peak load period, implicit increase of demand [3] , almost motionless vehicles. However it is reassuring to verify that the best algorithms can cope with this situation [4].

Both the conjugate gradient method and fixed point iteration behaved erratically for this large transit network.

[3] recall that demand is expressed in passengers per unit of time, not in absolute numbers.
[4] when $\mathcal{B}$ gets large μ^{k+1} approaches $-\Psi_l(\Phi^k)$.

$\mathcal{B}$	Profit Function							
	eqn(12)		eqn(12)$^{(\bullet)}$		eqn(16)		eqn(16)$^{(\bullet)}$	
	#it.	time$^{(1)}$	#it.	time$^{(1)}$	#it.	time$^{(1)}$	#it.	time$^{(1)}$
1000	6	240	5	150	9	260	6	210
2000	11	270	7	160	12	260	9	230
3000	27	580	10	210	14	270	13	280
4000	–	N.C.	14	300	18	340	17	340
5000	–	N.C.	20	360	23	420	21	440
6000	–	N.C.	24	470	29	480	27	460
7000	–	N.C.	38	600	34	570	35	590
8000	–	N.C.	52	800	39	600	41	690
9000	–	N.C.	69	1020	43	710	49	820
10000	–	N.C.	92	1390	48	720	52	810

$\mathcal{B}$ = fleet size; N.C.: time limit exceeded

(1) : microseconds

$(\bullet)$: function partially linearized.

Table 3a. Newton–Jacobi Iterates for partially linearized functions. (Small network)

$\mathcal{B}$	Profit Function					
	eqn(13)		eqn(14)		eqn(15)	
	#it.	time$^{(1)}$	#it.	time$^{(1)}$	#it.	time$^{(1)}$
1000	9	280	6	260	6	210
2000	12	290	9	240	9	260
3000	14	290	13	230	12	260
4000	17	320	18	400	17	330
5000	21	390	23	340	21	360
6000	27	570	30	460	28	420
7000	32	550	36	580	35	610
8000	36	650	42	640	41	660
9000	41	650	45	710	45	740
10000	46	700	46	650	50	820

$\mathcal{B}$ = fleet size; N.C.: time limit exceded

(1) : microseconds.

Table 3b. Newton–Jacobi Iterates for small network.

$\mathcal{B}$	Profit Function							
	eqn(15)				eqn(16)			
	$\mu^{(1)}$		$\mu^{(2)}$		$\mu^{(1)}$		$\mu^{(2)}$	
	#iter.	time$^{(3)}$	#iter.	time$^{(3)}$	#iter.	time$^{(3)}$	#iter.	time$^{(3)}$
1000	12	6200	3	3290	12	7020	3	3270
3000	11	6500	3	3280	11	6610	3	3200
5000	10	5660	3	3270	10	6020	3	3130
7000	10	5800	3	3300	10	6200	3	3230
9000	10	5450	3	3290	10	5800	3	3210
11000	9	6210	4	3520	9	6220	4	3590
13000	9	5340	4	3640	9	6110	4	3470
15000	9	5250	4	3610	11	7000	4	3520
17000	9	5910	4	3560	9	5960	4	3460
19000	10	6110	5	3940	10	6190	5	3890

$\mathcal{B}$ = fleet size

$^{(3)}$: microseconds

$\mu^{(1)}$: Newton's method applied to eqn(7)

$\mu^{(2)}$: estimated from eqn(9) .

Table 4. Results for the Santiago Network (Newton–Jacobi).

5. Conclusion.

In this paper we showed how to solve efficiently an equilibrium problem arising in a public transport deregulation context. The numerical tests demonstrated that it was worth investigating algorithms other than the "natural" fixed point iteration, both from the convergence and the computational points of view. It is interesting to notice that for the large problem it proved more efficient to use eqn(9) than eqn(7) to estimate the average profit μ, contrary to the situation arising in the small network. This can be explained by the fact that bus–induced congestion is much less important (relatively speaking) in the large Santiago network, thus making an adjustment such as the one provided by eqn(9) very "sensible". Finally let us mention that the interpretation of the results corresponding to very high values of $\mathcal{B}$ in the small example is delicate.

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